Winding Numbers

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This is a lecture note for the topology course [a] on Nov. 25, 2021. In this note, we study ***winding numbers***, which are fundamental objects in algebraic topology. We introduce its motivation, definition, simple properties, and use it to prove a toy theorem: the two-dimensional case of Brouwer fixed-point theorem.

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Our main subjects are *loops*, so we first define it formally.

Definition 0.1. A continuous map $\gamma : [0,1] \to \mathbb{C}$ is called a ***path***; it is called a ***loop*** if $\gamma(0) = \gamma(1)$.

Equivalently, we may write $\gamma : S^1 \to \mathbb{C}$, where S^1 is the circle that can be specified as $S^1 \overset{\text{def}}{=} \{x \in \mathbb{C} : |x| = 1\}$.

Note that a path γ cannot fill the entire complex plane, i.e., $\mathtt{Im}(\gamma) \neq \mathbb{C}$. This is because $[0,1]$ is compact (since it is closed and bounded) while $\mathbb{C}$ is not, and any continuous image of a compact space is compact. Without loss of generality, we may assume $0 \notin \mathtt{Im}(\gamma)$, and write $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ directly.

1 How many times does a path rotate around a point?

Let us begin with two paths in fig. 1. One might say that γ_1 'rotates more' than γ_2. How can we make sense of such intuition?

Consider the simplest loop $e_n : x \mapsto e^{2\pi i n x}$. Intuitively, e_n rotate n times (around 0) when x changes from 0 to 1. This can be generalized a little: if a loop γ can be written as

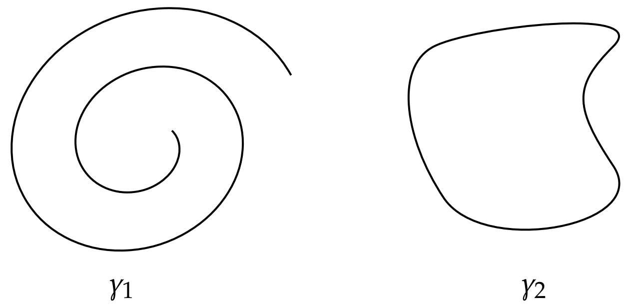


Figure 1: Two paths

$\exp \circ g$ for some continuous g, then $g(x)/i$ is the rotation angle at time x, so it is reasonable to define the winding number of γ as $(g(1) - g(0))/(2\pi i)$. Formally,

Definition 1.1. A continuous map $f : X \to \mathbb{C} \setminus \{0\}$ is an ***exponential*** if there exists a continuous $g : X \to \mathbb{C}$ such that $f = \exp \circ g$.

Two natural questions arise:

1. Are there other characterizations of being an exponential?
2. Is every path γ an exponential? If it is, we can define winding number as we wanted.

The rest of this section shall give affirmative answers to both questions and finally arrive at a formal definition of winding number.

Lemma 1.1. *Let $f : X \to \mathbb{C} \setminus \{0\}$ be a continuous map. If f never takes negative real values, then f is an exponential.*

Proof. Let $\mathbb{R}_{<0}$ denote the set of negative real numbers. Define a function $\texttt{LOG} : \mathbb{C} \setminus \mathbb{R}_{<0} \to \mathbb{C}$ via

$$\texttt{LOG}(z) \overset{\text{def}}{=} \log |z| + i\texttt{Arg}(z),$$

where $\texttt{Arg}(z) \in (\pi, \pi]$ is the principal value of $\texttt{arg}(z)$. It is known that $\texttt{LOG}$ is continuous on $\mathbb{C} \setminus \mathbb{R}_{<0}$. Since f takes values on $\mathbb{C} \setminus \mathbb{R}_{<0}$, we have $f = \exp \circ \texttt{LOG} \circ f$, and hence f is an exponential. □

Lemma 1.2. *If a continuous map $f : X \to \mathbb{C} \setminus \{0\}$ is an exponential, then it is null-homotopic.*

Proof. Suppose that $f = \exp \circ g$ for some continuous map g. Consider the homotopy

$$h : [0,1] \times X \to \mathbb{C}, (t, x) \mapsto \exp\left((1-t)g(x)\right). \tag{1}$$

That is, $h_t = \exp \circ (1-t)g$. Clearly, $h_0 = f$ and $h_1 = \mathbf{1}$ is a constant map. □

In fact, we shall prove that the converse of lemma 1.2 is also true, which gives us a necessary and sufficient condition of being an exponential. To reach that, we need the following theorem.

Theorem 1.3 (Rouche theorem)**.** *Let f_0 and f_1 be two continuous maps from X to $\mathbb{C} \setminus \{0\}$. If*

$$|f_0(x) - f_1(x)| < |f_0(x)| + |f_1(x)|, \forall x \in X, \tag{2}$$

then f_0/f_1 is an exponential.

Proof. We rewrite eq. (2) as

$$\left|\frac{f_0(x)}{f_1(x)} - 1\right| < \left|\frac{f_0(x)}{f_1(x)}\right| + 1, \forall x \in X. \tag{3}$$

Note that for any $x \in \mathbb{R}_{<0}$, we have $|x-1| = |x| + 1$, and hence eq. (3) implies $f_0(x)/f_1(x) \notin \mathbb{R}_{<0}$ for all $x \in X$. By lemma 1.1, we conclude that f_0/f_1 is an exponential. □

Proposition 1.1. Let f_0 and f_1 be two continuous maps from X to $\mathbb{C} \setminus \{0\}$. Then

$$f_0 \text{ and } f_1 \text{ are homotopic} \iff f_0/f_1 \text{ is an exponential.}$$

Proof. ($\Leftarrow$) If f_0/f_1 is an exponential, by lemma 1.2, there exists a homotopy h such that $h_0 = f_0/f_1$ and $h_1 = \mathbf{1}$. Then the homotopy $\widetilde{h}$ defined via $\widetilde{h}_t \overset{\text{def}}{=} h_t \cdot f_1$ is a homotopy from f_0 to f_1.

($\Rightarrow$) Suppose that there exists a homotopy h such that $h_0 = f_0/f_1$ and $h_1 = \mathbf{1}$. Choose a large enough number $N \in \mathbb{N}$. For $i \in [N] \cup \{0\}$, define $\widetilde{h}_i \overset{\text{def}}{=} h_{i/N}$. By definition, $\widetilde{h}_0 = h_0 = f_0/f_1, \widetilde{h}_N = h_1 = \mathbf{1}$, and thus

$$\frac{f_0}{f_1} = \widetilde{h}_0 = \frac{\widetilde{h}_0}{\widetilde{h}_1} \cdot \frac{\widetilde{h}_1}{\widetilde{h}_2} \cdots \frac{\widetilde{h}_{N-1}}{\widetilde{h}_N}.$$

It suffices to show that for every $i \in \{0, 1, \ldots, N-1\}$, $\widetilde{h}_i / \widetilde{h}_{i+1}$ is an exponential. By Hausdorff property, there exists $\varepsilon > 0$ such that $\left|\widetilde{h}_i(x)\right| > \varepsilon$ for all x and i. We choose a big N such that $\left|\widetilde{h}_{i+1}(x) - \widetilde{h}_i(x)\right| < 2\varepsilon$, then we get

$$\left|\widetilde{h}_{i+1}(x) - \widetilde{h}_i(x)\right| < 2\varepsilon < \left|\widetilde{h}_{i+1}(x)\right| + \left|\widetilde{h}_i(x)\right|, \forall x \in X, \forall i.$$

By theorem 1.3, $\widetilde{h}_i / \widetilde{h}_{i+1}$ is an exponential for every i; this finishes the proof. □

Proposition 1.2. A map $f : X \to \mathbb{C}$ is an exponential iff f is null-homotopic.

Proof. This immediately follows from proposition 1.1 by setting $f_0 = f, f_1 = \mathbf{1}$. □

Proposition 1.3. Every path $\gamma : [0, 1] \to \mathbb{C} \setminus \{0\}$ is an exponential.

Proof. Consider the homotopy $h : [0, 1] \times [0, 1] \to \mathbb{C}, (t, x) \mapsto \gamma((1-t)x)$. That is, γ is null homotopic; by proposition 1.2, we conclude that it is an exponential. □

Finally, we are ready to give the formal definition of winding number.

Definition 1.2 (Winding number)**.** Let $\gamma : [0, 1] \to \mathbb{C} \setminus \{0\}$ be a loop. By proposition 1.3, $\gamma = \exp \circ g$ for some $g : [0, 1] \to \mathbb{C}$. The ***winding number of*** γ ***at 0*** is defined as

$$\mathtt{WN}(\gamma) \stackrel{\text{def}}{=} \frac{g(1) - g(0)}{2\pi i} \in \mathbb{Z}.$$

Moreover, if $\gamma : [0, 1] \to \mathbb{C} \setminus \{\mathsf{pt}\}$, i.e., γ does not pass the point pt, the winding number of γ at pt is defined as

$$\mathtt{WN}(\gamma; \mathsf{pt}) \stackrel{\text{def}}{=} (\gamma - \mathsf{pt}).$$

One piece is missing in the above definition. What if we have two different functions g_1, g_2 such that $\gamma = \exp \circ g_1 = \exp \circ g_2$? We have to argue that g_1 and g_2 give the same winding number for γ, so that $\mathtt{WN}(\gamma)$ is well-defined. Since $\gamma(t) = \exp(g_1(t)) = \exp(g_2(t))$, we have $\phi(t) \stackrel{\text{def}}{=} g_1(t) - g_2(t) = n \cdot 2\pi i$ for some $n \in \mathbb{Z}$. Note that ϕ is continuous, it must be a constant; that is, $g_1(t) - g_2(t)$ does not depend on t. Hence, $g_1(1) - g_2(1) = g_1(0) - g_2(0)$, or equivalently

$$g_1(1) - g_1(0) = g_2(1) - g_2(0),$$

which means $\mathtt{WN}(\gamma)$ is well-defined.

2 Properties and applications

We say a homotopy $h : [0,1] \times [0,1] \to \mathbb{C} \setminus \{\mathsf{pt}\}$ is ***loop-preserving*** if at every time $t \in [0,1]$, the map $h_t(\cdot) \stackrel{\text{def}}{=} h(t, \cdot)$ is a loop.

Lemma 2.1. *Let $\gamma : [0,1] \to \mathbb{C} \setminus \{\mathsf{pt}\}$ be a loop. Then*

$$\mathtt{WN}(\gamma) = 0 \iff \gamma \text{ is null homotopic and the homotopy is loop-preserving.}$$

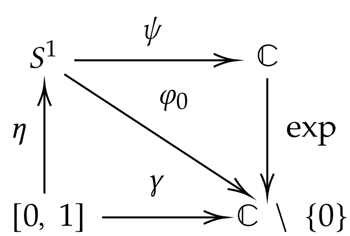
Proof. Without loss of generality, say $\mathsf{pt} = 0$.

($\Rightarrow$) $\mathtt{WN}(\gamma) = 0$ means $\gamma = \exp \circ g$ for some g with $g(0) = g(1)$. Then the homotopy $h_t(x) \stackrel{\text{def}}{=} \exp\left((1-t)g(x)\right)$ is loop-preserving and $h_1 = \mathbf{1}$.

($\Leftarrow$)Let h be a loop-preserving homotopy with $h_0 = \gamma, h_1 = \mathbf{1}$. At any time $t \in [0,1]$, h_t is loop, so h_t can be decomposed as $h_t = \varphi_t \circ \eta$, where φ_t is a continuous map from S^1 to $\mathbb{C} \setminus \{0\}$, and

$$\eta : [0,1] \to S^1, x \mapsto \exp(2\pi x i).$$

Note that $(\varphi_t)_{t \in [0,1]}$ forms a homotopy from S^1 to $\mathbb{C} \setminus \{0\}$, where $\varphi_1 = \mathbf{1}$. By lemma 1.2, φ_0 is an exponential, i.e., $\varphi_0 = \exp \circ \psi$ for some continuous map $\psi : S^1 \to \mathbb{C}$. Since $\gamma = h_0 = \varphi_0 \circ \eta = \exp \circ \psi \circ \eta$, the relation between these maps is demonstrated in the following commutative diagram.



Write $g \stackrel{\text{def}}{=} \psi \circ \eta$, and we have

$$\eta(0) = \eta(1) \implies g(1) = g(0) \implies \mathtt{WN}(\gamma) = 0.$$

$\square$

Lemma 2.2. *Let γ_1 and γ_2 be two loops. Then $\mathtt{WN}(\gamma) = \mathtt{WN}(\gamma_1) + \mathtt{WN}(\gamma_2)$, where $\gamma \stackrel{\text{def}}{=} \gamma_1 \cdot \gamma_2$.*

Proof. Suppose that $\gamma_1 = \exp \circ g_1, \gamma_2 = \exp \circ g_2$, then $\gamma = \exp \circ g$ where $g = g_1 + g_2$. By definition,

$$\mathtt{WN}(\gamma) = \frac{g(1) - g(0)}{2\pi i} = \frac{g_1(1) - g_1(0)}{2\pi i} + \frac{g_2(1) - g_2(0)}{2\pi i} = \mathtt{WN}(\gamma_1) + \mathtt{WN}(\gamma_2).$$

□

Proposition 2.1. For any loop γ,

$$\mathtt{WN}(\gamma) = n \iff \gamma \text{ is homotopic to } e_n \text{ and the homotopy is loop-preserving,}$$

where we recall that $e_n : x \mapsto e^{2n\pi ix}$.

Proof. Let γ be a loop. Since $\mathtt{WN}(e_n) = n$, by lemma 2.2, we have

$$\mathtt{WN}(\gamma) = n \iff \mathtt{WN}(\gamma \cdot e_{-n}) = 0.$$

According to lemma 2.1,

$$\begin{aligned} \mathtt{WN}(\gamma \cdot e_{-n}) = 0 &\iff \gamma \cdot e_{-n} \text{ is null homotopic and the homotopy is loop-preserving} \\ &\iff \gamma \text{ is homotopic to } e_n \text{ and the homotopy is loop-preserving.} \end{aligned}$$

□

A toy application: a special case of Brouwer fixed-point theorem

Finally, we use winding numbers to prove a special case of Brouwer fixed-point theorem: the 2-dimensional case:

Theorem 2.3 (Brouwer fixed-point theorem, 2-dimensional case)**.** *Every continuous function from disc D^2 to itself has a fixed point.*

Here we specify the disc D^2 as

$$D^2 \stackrel{\text{def}}{=} \{x \in \mathbb{C} : |x| \leq 1\}.$$

We shall draw on the following lemma:

Lemma 2.4. *There is no continuous map $f : D^2 \to S^1$ such that $f|_{S^1}$ is identity map, i.e., f is invariant on S^1.*

Proof. Assume toward contradiction that $f : D^2 \to S^1$ is a continuous map with $f|_{S^1} = \mathtt{Id}_{S^1}$. Define the map

$$h : [0,1] \times [0,1], (t,x) \mapsto f\left((1-t)\exp(2\pi x i)\right).$$

As usual, we write $h_t(\cdot) \overset{\text{def}}{=} h(t,\cdot)$. Since f is invariant on S^1, we have $h_0 = e_1$; and h_1 is a constant map always equals to $f(0)$. That is, e_1 is null-homotopic and the homotopy h is clearly loop-preserving. By lemma 2.1, $\mathtt{WN}(e_1) = 0$, but by definition $\mathtt{WN}(e_1) = 1$, a contradiction. □

Proof of theorem 2.3. Assume toward contradiction that there is a continuous map $f : D^2 \to D^2$ with no fixed point. For $x \in D^2$, since $f(x) \neq x$, the ray

$$\ell_x \overset{\text{def}}{=} \{f(x) + \lambda(x - f(x)) : \lambda \geq 0\}$$

intersects with S^1 at exactly one point, as is shown in fig. 2. We denote this single point as $\phi(x)$. Note that $\phi : D^2 \to S^1$ is continuous as long as f is, and ϕ is invariant on S^1, which is in contradiction with lemma 2.4. □

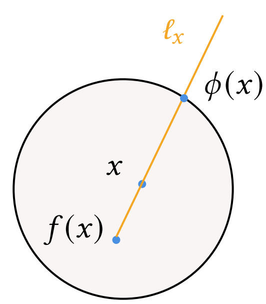


Figure 2: The definition of $\phi(x)$.